

The Roller-Coaster Conjecture Revisited

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Abstract

A graph is *well-covered* if all its maximal independent sets are of the same cardinality [25]. If G is a well-covered graph, has at least two vertices, and $G - v$ is well-covered for every vertex v , then G is a *1-well-covered graph* [26]. We call G a *λ -quasi-regularizable graph* if $\lambda \cdot |S| \leq |N(S)|$ for every independent set S of G . The *independence polynomial* $I(G; x)$ is the generating function of independent sets in a graph G [9].

The Roller-Coaster Conjecture [24], saying that for every permutation σ of the set $\{\lceil \frac{\alpha}{2} \rceil, \dots, \alpha\}$ there exists a well-covered graph G with independence number α such that the coefficients (s_k) of $I(G; x)$ satisfy

$$s_{\sigma(\lceil \frac{\alpha}{2} \rceil)} < s_{\sigma(\lceil \frac{\alpha}{2} \rceil + 1)} < \dots < s_{\sigma(\alpha)},$$

has been validated in [6].

In this paper we show that independence polynomials of λ -quasi-regularizable graphs are partially unimodal. More precisely, the coefficients of an upper part of $I(G; x)$ are in non-increasing order. Based on this finding, we prove that the domain of the Roller-Coaster Conjecture can be shortened up to:

$$\left\{ \left\lceil \frac{\alpha}{2} \right\rceil, \left\lfloor \frac{\alpha}{2} \right\rfloor + 1, \dots, \min \left\{ \alpha, \left\lceil \frac{n-1}{3} \right\rceil \right\} \right\}$$

for well-covered graphs, and up to

$$\left\{ \left\lceil \frac{2\alpha}{3} \right\rceil, \left\lfloor \frac{2\alpha}{3} \right\rfloor + 1, \dots, \min \left\{ \alpha, \left\lceil \frac{n-1}{3} \right\rceil \right\} \right\}$$

for 1-well-covered graphs, where α stands for the independence number, and n is the cardinality of the vertex set.

Keywords: independent set, well-covered graph, 1-well-covered graph, corona of graphs, independence polynomial.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G) \neq \emptyset$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G induced by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. We also denote by $G - F$ the subgraph of G obtained by deleting the edges of F , for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$.

The *neighborhood* $N(v)$ of $v \in V(G)$ is the set $\{w : w \in V(G) \text{ and } vw \in E(G)\}$, while the *closed neighborhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

C_n, K_n, P_n denote respectively, the cycle on $n \geq 3$ vertices, the complete graph on $n \geq 1$ vertices, and the path on $n \geq 1$ vertices.

The *disjoint union* of the graphs G_1, G_2 is the graph $G_1 \cup G_2$ having the disjoint unions $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$ as a vertex set and an edge set, respectively. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G .

An *independent* set in G is a set of pairwise non-adjacent vertices. An independent set of maximum size is a *maximum independent set* of G , and the *independence number* of G , denoted $\alpha(G)$, is the cardinality of a maximum independent set in G .

A graph is *well-covered* if all its maximal independent sets are of the same size [25]. If G is well-covered, without isolated vertices, and $|V(G)| = 2\alpha(G)$, then G is a *very well-covered graph* [7]. The only well-covered cycles are C_3, C_4, C_5 and C_7 , while C_4 is the only very well-covered cycle. A well-covered graph (with at least two vertices) is *1-well-covered* if the deletion of every vertex of the graph leaves a graph, which is well-covered as well [26]. For instance, K_2 is 1-well-covered, while P_4 is very well-covered, but not 1-well-covered. Notice that C_7 is well-covered but not 1-well-covered. The only 1-well-covered cycles are C_3 and C_5 . A graph G belongs to *class* W_2 if every two disjoint independent sets in G are contained in two disjoint maximum independent sets [26, 27]. Clearly, $W_1 \supseteq W_2$, where W_1 is the family of all well-covered graphs.

Theorem 1.1 [26] *Let G have no isolated vertices. Then G is 1-well-covered if and only if G belongs to the class W_2 .*

If G has an isolated vertex, then it is contained in all maximum independent sets, and hence G cannot be in class W_2 . However, a graph having isolated vertices may be 1-well-covered; e.g., $K_3 \cup K_1$.

Theorem 1.2 [21] *Let G be a graph without isolated vertices. Then G is 1-well-covered if and only if for each non-maximum independent set A there are at least two disjoint independent sets B_1, B_2 such that $A \cup B_1, A \cup B_2$ are maximum independent sets in G .*

Let s_k be the number of independent sets of size k in a graph G . The polynomial

$$I(G; x) = s_0 + s_1x + s_2x^2 + \cdots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the *independence polynomial* of G [9]. For a survey on independence polynomials of graphs see [14]. Closed formulae for $I(G; x)$ of several families of graphs one can find in [19, 31], while some factorizations of independence polynomials for certain classes of graphs are given in [29].

A polynomial is called unimodal if the sequence $(a_0, a_1, a_2, \dots, a_n)$ of its coefficients is *unimodal*, i.e., if there exists an index $k \in \{0, 1, \dots, n\}$, such that

$$a_0 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n.$$

In [1] it is proved that for every permutation σ of $\{1, 2, \dots, \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that the coefficients of $I(G; x)$ satisfy $s_{\sigma(1)} < s_{\sigma(2)} < \dots < s_{\sigma(\alpha)}$.

Theorem 1.3 [18, 24] *If G is a well-covered graph, then $s_0 \leq s_1 \leq \dots \leq s_{\lceil \frac{\alpha(G)}{2} \rceil}$.*

Several results concerning the independence polynomials of well-covered graphs are presented in [4, 11, 12, 13]. It is known that there exist well-covered graphs whose independence polynomials are not unimodal [16, 24].

Conjecture 1.4 (Roller-Coaster Conjecture) [24] *For every permutation σ of the set $\{\lceil \frac{\alpha}{2} \rceil, \dots, \alpha\}$ there is a well-covered graph G with $\alpha(G) = \alpha$ such that the coefficients of $I(G; x)$ satisfy $s_{\sigma(\lceil \frac{\alpha}{2} \rceil)} < s_{\sigma(\lceil \frac{\alpha}{2} \rceil + 1)} < \dots < s_{\sigma(\alpha)}$.*

The Roller-Coaster Conjecture has been verified for well-covered graphs G having $\alpha(G) \leq 7$ [24], and later for $\alpha(G) \leq 11$ [23]. In the case of very well-covered graphs, the domain of the Roller-Coaster Conjecture can be shortened to $\{\lceil \frac{\alpha}{2} \rceil, \lceil \frac{\alpha}{2} \rceil + 1, \dots, \lceil \frac{2\alpha-1}{3} \rceil\}$, where α stands for the independence number [15]. Recently, the Roller-Coaster Conjecture was validated in [6].

In this paper we show that the domain of the Roller-Coaster Conjecture can be shortened to:

- $\{\lceil \frac{\alpha}{2} \rceil, \lfloor \frac{\alpha}{2} \rfloor + 1, \dots, \min\{\alpha, \lceil \frac{n-1}{3} \rceil\}\}$ for well-covered graphs of order n ;
- $\{\lceil \frac{2\alpha}{3} \rceil, \lceil \frac{2\alpha}{3} \rceil + 1, \dots, \min\{\alpha, \lceil \frac{n-1}{3} \rceil\}\}$ for 1-well-covered graphs of order n .

Actually, $\min\{\alpha, \lceil \frac{n-1}{3} \rceil\} < \alpha$ only for $n \leq 3\alpha - 2$. It means that one may formulate an overhauled Roller-Coaster Conjecture as follows.

Conjecture 1.5 *Let $\alpha \geq 2$ and $n \geq 4$ be integers satisfying $2\alpha \leq n \leq 3\alpha - 2$. Then for every permutation σ of the set $\{\lceil \frac{\alpha}{2} \rceil, \lceil \frac{\alpha}{2} \rceil + 1, \dots, \lceil \frac{n-1}{3} \rceil\}$ there exists a well-covered graph G with $\alpha(G) = \alpha$ and $|V(G)| = n$ such that the coefficients of $I(G; x)$ satisfy $s_{\sigma(\lceil \frac{\alpha}{2} \rceil)} < s_{\sigma(\lceil \frac{\alpha}{2} \rceil + 1)} < \dots < s_{\sigma(\lceil \frac{n-1}{3} \rceil)}$.*

2 Results

We call G a λ -quasi-regularizable graph if $\lambda > 0$ and $\lambda \cdot |S| \leq |N(S)|$ is true for every independent set S of G . If $\lambda = 1$, then G is a quasi-regularizable graph [3].

For a graph G and $1 \leq k < \alpha(G)$, let $\Omega_k(G) = \{W : |W| = k, W \text{ is independent in } G\}$ and $H_k(G) = (\Omega_k(G), \Omega_{k+1}(G), Y)$ be the bipartite graph with bipartition $\{\Omega_k(G), \Omega_{k+1}(G)\}$ and such that $WU \in Y$ if and only if $W \subset U$. It is clear that $|\Omega_k| = s_k$.

Theorem 2.1 *If G is a λ -quasi-regularizable graph of order n , then the following assertions are true:*

- (i) $(k+1) \cdot s_{k+1} \leq (n - (\lambda+1)k) \cdot s_k, 0 \leq k < \alpha(G);$
- (ii) $s_r \geq s_{r+1} \geq \dots \geq s_{\alpha(G)}, \text{ for } r = \left\lceil \frac{n-1}{\lambda+2} \right\rceil.$

Proof. Every $U \in \Omega_{k+1}(G)$ has $k+1$ subsets in $\Omega_k(G)$, which means that the degree of every vertex U in $H_k(G)$ is equal to $k+1$. Consequently, we obtain

$$|Y| = (k+1) \cdot |\Omega_{k+1}(G)| = (k+1) \cdot s_{k+1}.$$

Every $W \in \Omega_k(G)$ may be extended to some $U \in \Omega_{k+1}(G)$ by means of a vertex belonging to $V(G) - N[W]$. Since G is a λ -quasi-regularizable, we have

$$|N[W]| = |W \cup N(W)| \geq (\lambda+1) \cdot |W|,$$

and hence,

$$(k+1) \cdot s_{k+1} \leq (n - (\lambda+1)k) \cdot s_k.$$

Therefore, we get

$$s_{k+1} \leq \frac{n - (\lambda+1)k}{k+1} \cdot s_k,$$

which implies $s_{k+1} \leq s_k$ for every k satisfying

$$\frac{n - (\lambda+1) \cdot k}{k+1} \leq 1 \Leftrightarrow k \geq \left\lceil \frac{n-1}{\lambda+2} \right\rceil,$$

as claimed. ■

In particular, for $\lambda = 1$, we deduce the following.

Corollary 2.2 *Let G be a quasi-regularizable graph of order $n \geq 2$ with $\alpha(G) = \alpha$. Then*

- (i) $(k+1) \cdot s_{k+1} \leq (n - 2k) \cdot s_k, 1 \leq k < \alpha;$
- (ii) $s_{\lceil \frac{n-1}{3} \rceil} \geq s_{\lceil \frac{n-1}{3} \rceil + 1} \geq \dots \geq s_\alpha.$

Taking into account Theorem 1.3, Corollary 2.2, and the fact that every well-covered graph is quasi-regularizable [3], we arrive at the following.

Corollary 2.3 *Let G be a well-covered graph of order $n \geq 2$ with $\alpha(G) = \alpha$. Then*

- (i) $(\alpha - k) \cdot s_k \leq (k+1) \cdot s_{k+1} \leq (n - 2k) \cdot s_k, 1 \leq k < \alpha;$
- (ii) $s_0 \leq s_1 \leq \dots \leq s_{\lceil \frac{\alpha}{2} \rceil}$ and $s_{\lceil \frac{n-1}{3} \rceil} \geq s_{\lceil \frac{n-1}{3} \rceil + 1} \geq \dots \geq s_\alpha.$

Combining Theorem 1.3 and Corollary 2.3, we infer that for well-covered graphs, the domain of the Roller-Coaster Conjecture can be shortened to $\{\lceil \frac{\alpha}{2} \rceil, \lceil \frac{\alpha}{2} \rceil + 1, \dots, \lceil \frac{n-1}{3} \rceil\}$, whenever $2 \leq \alpha$ and $4 \leq n \leq 3\alpha - 2$.

Since each very well-covered graph is of order twice its independence number, we obtain the following.

Corollary 2.4 [15] *If G is a very well-covered graph of order $n \geq 2$ with $\alpha(G) = \alpha$, then $s_0 \leq s_1 \leq \dots \leq s_{\lceil \frac{\alpha}{2} \rceil}$ and $s_{\lceil \frac{2\alpha-1}{3} \rceil} \geq s_{\lceil \frac{2\alpha-1}{3} \rceil + 1} \geq \dots \geq s_\alpha.$*

Clearly, nK_2 is 1-well-covered for $n \geq 1$, and has exactly $2\alpha(G)$ vertices, while each graph $G \in \{C_5 \cup nK_2, C_3 \cup nK_2\}, n \geq 1$, is 1-well-covered and has exactly $2\alpha(G) + 1$ vertices. One can show that C_3 and C_5 are the only connected 1-well-covered graphs with exactly $2\alpha(G) + 1$ vertices [21].

Proposition 2.5 [21] *If a connected graph $G \neq K_2$ is 1-well-covered, then:*

- (i) G has at least $2\alpha(G) + 1$ vertices;
- (ii) $|A| < |N(A)|$ for every independent set A .

Proposition 2.5(i) implies that K_2 is the unique very well-covered connected graph and also 1-well-covered. In addition, $I(K_2; x) = 1 + 2x$ is unimodal.

Theorem 2.6 *If G is a connected 1-well-covered graph, $|V(G)| = n > 2$, and $\alpha(G) = \alpha$, then the following assertions are true:*

- (i) $2(\alpha - k) \cdot s_k \leq (k + 1) \cdot s_{k+1}, 1 \leq k < \alpha$;
- (ii) $s_0 \leq s_1 \leq \dots \leq s_{\lceil \frac{2\alpha}{3} \rceil}$;
- (iii) $(k + 1) \cdot s_{k+1} < (n - 2k) \cdot s_k, 1 \leq k < \alpha$;
- (iv) $s_{\lceil \frac{n-1}{3} \rceil} > s_{\lceil \frac{n-1}{3} \rceil + 1} > \dots > s_\alpha$.

Proof. (i) According to Proposition 2.5(i), we have that $2\alpha \cdot s_0 = 2\alpha \leq s_1 = |V(G)|$.

Every $U \in \Omega_{k+1}(G)$ has $k+1$ subsets in $\Omega_k(G)$, which means that the degree of every vertex U in H is equal to $k+1$. Consequently, $|Y| = (k+1) \cdot |\Omega_{k+1}(G)| = (k+1) \cdot s_{k+1}$. On the other hand, by Theorem 1.2, every $W \in \Omega_k(G)$ can be extended by two disjoint independent sets B_1, B_2 such that $W_i \cup B_1, W_i \cup B_2$ are maximum independent sets in G . In other words, the degree of every vertex $W \in \Omega_k(G)$ is at least $2(\alpha - k)$.

In conclusion, we obtain $2(\alpha - k) \cdot s_k \leq (k + 1) \cdot s_{k+1}$, and this implies (i).

(ii) According Part (i), we have

$$s_k \leq \frac{k+1}{2(\alpha-k)} \cdot s_{k+1},$$

which ensures that $s_k \leq s_{k+1}$ for every k satisfying $\frac{k+1}{2(\alpha-k)} \leq 1$, i.e., for $k \leq \frac{2\alpha-1}{3}$, at least. In other words, the monotone part of the sequence of coefficients goes up to $k+1 \leq \lfloor \frac{2\alpha+2}{3} \rfloor = \lceil \frac{2\alpha}{3} \rceil$.

(iii) and (iv) By Proposition 2.5(ii), $|A| < |N(A)|$ for every independent set A . To get the result, one has just to follow the lines of the proof of Theorem 2.1 changing “ \leq ” for “ $<$ ”, when needed. ■

In other words, for 1-well-covered graphs, the domain of the Roller-Coaster Conjecture can be shortened to $\{\lceil \frac{2\alpha}{3} \rceil, \lceil \frac{2\alpha}{3} \rceil + 1, \dots, \lceil \frac{n-1}{3} \rceil\}$, whenever $n \leq 3\alpha - 2$.

Let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of graphs indexed by the vertex set of a graph G . The corona $G \circ \mathcal{H}$ of G and \mathcal{H} is the disjoint union of G and $H_v, v \in V(G)$, with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v . If $H_v = H$ for every $v \in V(G)$, then we denote $G \circ H$ instead of $G \circ \mathcal{H}$ [8].

Theorem 2.7 [21] *Let G be an arbitrary graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of non-empty graphs. Then $G \circ \mathcal{H}$ is 1-well-covered if and only if each $H_v \in \mathcal{H}$ is a complete graph of order two at least, for every non-isolated vertex v , while for each isolated vertex u , its corresponding H_u may be any complete graph.*

It is easy to see that $H \circ K_1$ is very well-covered for every graph H , and some properties of $I(H \circ K_1; x)$ are presented in [18]. Several findings concerning the palindromicity of $I(H \circ Y; x)$ are proved in [17, ?, 30].

Theorem 2.8 [10] $I(H \circ Y; x) = (I(Y; x))^n \bullet I\left(H; \frac{x}{I(Y; x)}\right)$, where $n = |V(H)|$.

Theorem 2.8 allows finding closed formulae for $I(H \circ Y; x)$, once such formulae are known for both $I(H; x)$ and $I(Y; x)$; for instance, one can obtain closed formulae for $I(H \circ K_p; x)$, where $H \in \{P_n, C_n, K_{1,n}\}$ [2, 9, 18].

Theorem 2.9 Let H be a connected graph. If $G = H \circ K_2$ and $\alpha(G) = \alpha$, then the following assertions are true:

- (i) G is a 1-well-covered graph;
- (ii) G is a 2-quasi-regularizable graph of order $n = 3\alpha$;
- (iii) $2(\alpha - k) \cdot s_k \leq (k + 1) \cdot s_{k+1} \leq 3(\alpha - k) \cdot s_k, 1 \leq k < \alpha$;
- (iv) $s_0 \leq s_1 \leq \dots \leq s_{\lceil \frac{2\alpha}{3} \rceil}$ and $s_{\lceil \frac{3\alpha-1}{4} \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$;
- (v) if $\alpha \geq 3$, then $s_{\alpha-3} \cdot s_{\alpha-1} \leq s_{\alpha-2}^2$ and $s_{\alpha-2} \cdot s_\alpha \leq s_{\alpha-1}^2$;
- (vi) if $\alpha \leq 17$, then $I(G; x)$ is unimodal.

Proof. (i) It follows from Theorem 2.7.

(ii) Let $S = S_1 \cup S_2$ be an independent set in G , where $S_1 \subseteq V(H)$, while $S_2 \subseteq V(G) - V(H)$. Then $2|S_1| = |N_G(S_1) - V(H)| \leq |N_G(S_1)|$, because every vertex of S_1 has exactly two neighbors in $V(G) - V(H)$, and $2|S_2| = |N_G(S_2)|$, since each vertex from S_2 has exactly two neighbors in G . Hence, we get that:

$$2|S| = 2|S_1| + 2|S_2| \leq |N_G(S_1) - V(H)| + |N_G(S_2)| \leq |N_G(S)|,$$

i.e., G is 2-quasi-regularizable. Clearly, $\alpha = |V(H)|$. Thus $n = 3\alpha$.

(iii) It follows from Theorem 2.6(i), Theorem 2.1(i), and the fact that $n = 3\alpha$.

(iv) By Theorem 2.1, $s_{\lceil \frac{3\alpha-1}{4} \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$, because G is 2-quasi-regularizable. According to Theorem 2.6(ii), the polynomial $I(G \circ K_2; x)$ satisfies $s_0 \leq s_1 \leq \dots \leq s_{\lceil \frac{2\alpha}{3} \rceil}$.

(v) Let us specialize the inequality $2(\alpha - k) \cdot s_k \leq (k + 1) \cdot s_{k+1}$ at $k = \alpha - 3$ and the inequality $(k + 1) \cdot s_{k+1} \leq 3(\alpha - k) \cdot s_k$ at $k = \alpha - 2$. It implies

$$s_{\alpha-3} \cdot s_{\alpha-1} \leq \frac{(\alpha - 2)}{(\alpha - 1)} \cdot s_{\alpha-2}^2 \leq s_{\alpha-2}^2$$

When we substitute $k = \alpha - 2$ and $k = \alpha - 1$ in the same manner, we obtain

$$s_{\alpha-2} \cdot s_\alpha \leq \frac{3(\alpha - 1)}{4\alpha} \cdot s_{\alpha-1}^2 \leq s_{\alpha-1}^2.$$

(vi) By part (iv), the sequence of coefficients of $I(G; x)$ is non-decreasing up to $\lceil \frac{2\alpha}{3} \rceil$ and non-increasing starting from $\lceil \frac{3\alpha-1}{4} \rceil$. In addition, the constraint $\alpha \leq 17$ ensures that $\lceil \frac{3\alpha-1}{4} \rceil - \lceil \frac{2\alpha}{3} \rceil \leq 1$. ■

In other words, if G can be represented as $H \circ K_2$, then G is 1-well-covered and the domain of the Roller-Coaster Conjecture can be shortened to $\{\lceil \frac{2\alpha}{3} \rceil, \lceil \frac{2\alpha}{3} \rceil + 1, \dots, \lceil \frac{3\alpha-1}{4} \rceil\}$.

It is known that:

- each polynomial with positive coefficients that has only real roots is unimodal;
- there exist graphs whose independence polynomials have all the roots real (for example, $K_{1,3}$ -free graphs [5], $P_n \circ K_1$ for any $n \geq 1$ [18]);
- $I(H \circ K_p; x)$ has only real roots if and only if the same is true for $I(H; x)$ [18, 22].

Hence, using Theorem 2.7, we get the following.

Corollary 2.10 *If $I(H; x)$ has only real roots and $p \geq 2$, then every graph*

$$G \in \{H \circ K_p, (H \circ K_p) \circ K_p, ((H \circ K_p) \circ K_p) \circ K_p, \dots\}$$

is 1-well-covered and its $I(G; x)$ is unimodal, as having all its roots real.

3 Conclusions and future work

In this paper we proved that for 1-well-covered graphs the “chaotic interval” $(\lceil \frac{\alpha}{2} \rceil, \lceil \frac{\alpha}{2} \rceil + 1, \dots, \alpha)$ involved in Roller-Coaster Conjecture can be shortened to $\{\lceil \frac{2\alpha}{3} \rceil, \lceil \frac{2\alpha}{3} \rceil + 1, \dots, \alpha\}$. Based on this finding, we propose a Roller-Coaster Conjecture for 1-well-covered graphs as follows.

Conjecture 3.1 *For every permutation σ of the set $\{\lceil \frac{2\alpha}{3} \rceil, \lceil \frac{2\alpha}{3} \rceil + 1, \dots, \alpha\}$ there exists a 1-well-covered graph G with $\alpha(G) = \alpha$ and $|V(G)| = n$ such that the coefficients of $I(G; x)$ satisfy $s_{\sigma(\lceil \frac{2\alpha}{3} \rceil)} < s_{\sigma(\lceil \frac{2\alpha}{3} \rceil + 1)} < \dots < s_{\sigma(\alpha)}$.*

We incline to think that Conjecture 3.1 can be validated using a technique similar to one presented in [6]. The only obstacle we see now is in constructing a 1-well-covered graph G such that for every given positive integer k each $S \in \Omega_{k+1}(G)$ is included in exactly two maximum independent sets.

Problem 3.2 *Characterize 1-well-covered graphs whose independence polynomials are unimodal.*

The nature and location of the roots of $I(G; x)$ for a well-covered graph G were first analyzed in [4]. It is worth mentioning that there are 1-well-covered graphs whose independence polynomials have non-real roots; e.g., $I(K_{1,3} \circ K_2; x) = 1 + 12x + 51x^2 + 93x^3 + 62x^4$. Taking into account Corollary 2.10, we propose the following.

Problem 3.3 *Characterize 1-well-covered graphs whose independence polynomials have all the roots real.*

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